Math 206A Lecture 11 Notes

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1 Polytopes and Permutahedra

1.1 Polytopes

Definition 1.1. A polytope $P \subseteq \mathbb{R}^d$ is either of the following two equivalent things:

1. $P = \operatorname{conv}(X)$, $|X| < \infty$, $X \subseteq \mathbb{R}^d$.
2. $P = \bigcap H_i$ such that P is compact, where the H_i are half spaces.

Definition 1.2. The **dimension** of P is $\dim(P) = \dim_{\mathbb{R}} \langle P \rangle$, the affine subspace of $\mathbb{R}^n$ spanned by P.

Definition 1.3. A **face** $F \subseteq P$ is a subset of P such that there exists an affine subspace $W \subseteq \mathbb{R}^d$ such that

1. $F = P \cap W$,
2. there exists a half-space H such that $P \subseteq H$, $W \subseteq \partial H$, and $P \cap \partial H = F$.[1]

Example 1.1. Let C_3 be the cube in $\mathbb{R}^3$. Then $C_3 = \operatorname{conv}(\{(\pm 1, \pm 1, \pm 1)\})$. On the other hand, $C_3 = \bigcap_{i=1}^3 \{x : x_i \leq 1\} \cap \bigcap_{i=1}^3 \{x : x_i \geq -1\}$. The faces are $C_3 \cap \{x : x_i = \pm 1\}$.

Definition 1.4. An **edge** is a 1-dimensional face. A **vertex** is a 0-dimensional face. A **facet** is a $(d-1)$-dimensional face.

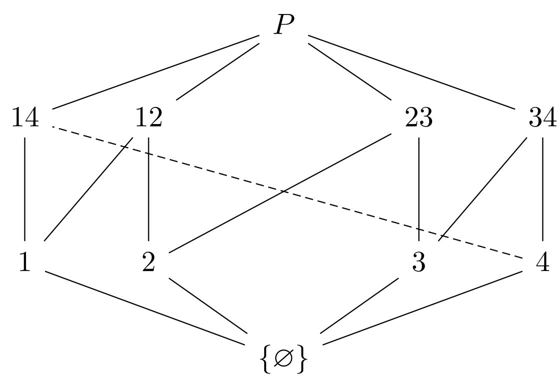
Definition 1.5. The **graph** $\Gamma(P)$ of a polytope is a graph $\Gamma = (V, E)$, where $V(P)$ is the set of vertices of P and $E(P)$ is the set of edges of P.

Definition 1.6. The **face lattice** $\alpha(P)$ is the partially ordered set of faces of P, ordered by inclusion.

[1] This second condition implies the first, so you should really think of it as a clarification of the previous condition.

This is a lattice because the meet of F and F' is $F \cap F'$, and the join of F and F' is $\langle F \cup F' \rangle \cap P$.

Example 1.2. Let P be a square in $\mathbb{R}^2$. Then $\Gamma(P)$ is the graph of the boundary of the square, and $\alpha(P)$ is



Here is a theorem we will prove later.

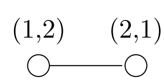
Theorem 1.1 (Blind-Mani)**.** *If $P \subseteq \mathbb{R}^d$ is simple, then Γ, the graph of P determines the face lattice of P.*

1.2 Permutahedra

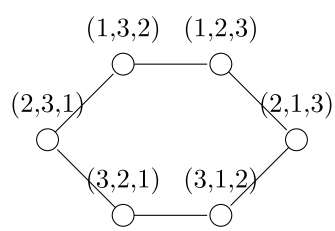
Definition 1.7. This **permutahedron** is $P = \mathrm{conv}(\{(\sigma(1), \dots, \sigma(n)) : \sigma \in S_n\})$.

Observe that $\dim(P) = n - 1$.

Example 1.3. For $n = 2$, the permutahedron is



For $n = 3$, we have



Proposition 1.1. *$\Gamma(P_n) \cong \mathrm{Cay}(S_n, R_n)$, where R_n is the set of transpositions $(i\ j)$ with $1 \leq i, j \leq n$ with a left action.*

Observe that P_n is simple. In particular, we can figure out the $\mathcal{F}$-vector. Consider a linear functional $\varphi : \mathbb{R}^n \to \mathbb{R}$ nonconstant on edges where

$$\varphi(x_1, \ldots, x_n) = x_1 + \varepsilon x_2 + \varepsilon^2 x_3 + \cdots + \varepsilon^{n-1} x_n.$$

Then $\operatorname{ind}(\sigma)$ is the number of $i \in \{1, \ldots, n-1\}$ such that $\varphi((i\ \ i+1)\sigma) > \varphi(\sigma)$. This is the number of i such that $\sigma^{-1}(i) < \sigma^{-1}(i+1)$. So $g_k = h_k^{(\varphi)}$ is the number of $\sigma \in S_N$ such that σ^{-1} has k ascents.

Let $A(n, K)$ be the number of $\sigma \in S_n$ with exactly k ascents.[2] We can prove the following proposition.

Proposition 1.2.

$$A(n, k) = (n - K)A(n-1, k-1) + (k+1)A(n-1, k)$$

Example 1.4. This is called the **Birkhoff polytope**. It is the set of matrices of non-negative entries such that the sum of the rows and columns are all 1. Formally, this is $B_n = \bigcap_{i=1}^n \bigcap_{j=1}^n \{x : x_{i,j} \geq 0\} \cap \bigcap_{j=1}^n \{x : \sum_{i=1}^n x_{i,j} = 1\} \cap \bigcap_{i=1}^n \{x : \sum_{j=1}^n x_{i,j} = 1\}$.

Theorem 1.2. $V(B_n) = \{\mathrm{Mat}(\sigma) : \sigma \in S_n\}$. $E(B_n) = \{(\sigma, w\sigma) : w \in S_n$ *is a cycle*$\}$

Corollary 1.1. $\deg_\Gamma(v)$ *is the number of cycles in* S_n*, and* $\dim(B_n) = (n-1)^2$.

Question: Is $f_i(B)$ computable in polynomial time?

Theorem 1.3 (Pak)**.** *Let* Q_n *be the set of such matrices but with dimension* $n \times (n+1)$*. Then* $f_i(Q_n)$ *can be computed in polynomial time.*

[2] The bivariate generating function for $A(n, k)$ has a nice form.